

# CLOSE-TO-EQUILIBRIUM REGULARITY FOR REACTION-DIFFUSION SYSTEMS

BAO QUOC TANG

**ABSTRACT.** The close-to-equilibrium regularity of solutions to a class of reaction-diffusion systems is investigated. The considered systems typically arise from chemical reaction networks and satisfy a complex balanced condition. Under some restrictions on spatial dimensions ( $d \leq 4$ ) and order of nonlinearities ( $\mu = 1 + 4/d$ ), we show that if the initial data is close to a complex balanced equilibrium in  $L^2$ -norm, then classical solutions are shown global and converging exponentially to equilibrium in  $L^\infty$ -norm. Possible extensions to higher dimensions and order of nonlinearities are also discussed. The results of this paper improve the recent work [M.J. Cáceres and J.A. Cañizo, *Nonlinear Analysis* (2017), in press].

**Keywords:** Reaction-diffusion systems; Close-to-equilibrium regularity; Chemical reaction networks; Complex balanced condition.

## CONTENTS

1. Introduction and main results	1
2. Complex balanced chemical reaction networks	5
3. Spectral gap for the linearised operator	7
4. Proof of Theorem 1.1	10
5. Extension to higher dimensions	15
References	18

## 1. INTRODUCTION AND MAIN RESULTS

This paper deals with global existence of classical solutions to a class of reaction-diffusion systems with initial data being close to an equilibrium. The considered systems typically arise from chemical reaction networks. More precisely, we consider  $N$  chemical substances  $S_1, \dots, S_N$ , which react in the  $R$  reactions where the  $r$ -th reaction has the form

$$(1.1) \quad y_{r,1}S_1 + \dots + y_{r,N}S_N \xrightarrow{k_r} y'_{r,1}S_1 + \dots + y'_{r,N}S_N$$

for  $r = 1, \dots, R$ . Here  $y_r = (y_{r,1}, \dots, y_{r,N})$ ,  $y'_r = (y'_{r,1}, \dots, y'_{r,N}) \in (\{0\} \cup [1, +\infty))^N$  are stoichiometric coefficients, and  $k_r > 0$  is the reaction rate constant. We will assume naturally that there exists at least one  $i \in \{1, \dots, N\}$  such that  $y_{r,i} \neq y'_{r,i}$ . With a slight abuse of notation we will rewrite the reaction (1.1) as

$$y_r \xrightarrow{k_r} y'_r$$

---

*Date:* April 6, 2017.

in which  $y_r$  and  $y'_r$  are called *complexes* with  $y_r$  is the reactant and  $y'_r$  is the product of the corresponding reaction. Denote by  $\mathcal{C} = \{y_r, y'_r : r = 1, \dots, R\}$  the set of all complexes. Note that each complex  $y \in \mathcal{C}$  can be a reactant, a product or both (in possibly different reactions).

To set up a reaction-diffusion system modelling (1.1) we assume that the reactions take place in a bounded vessel  $\Omega \subset \mathbb{R}^d$  with smooth boundary  $\partial\Omega$  (e.g.  $C^{2+\epsilon}$  for  $\epsilon > 0$ ). For  $i = 1, \dots, N$ , denote by  $u_i(x, t)$  the concentration  $S_i$  at position  $x \in \Omega$  and at time  $t \geq 0$ . Assume moreover that each substance  $S_i$  diffuses at a positive constant rate  $d_i > 0$ . The corresponding reaction-diffusion system for  $u = (u_1, \dots, u_N)$  reads as

$$(1.2) \quad \begin{aligned} \partial_t u_i - d_i \Delta u_i &= f_i(u), & x \in \Omega, & \quad t > 0, \\ \nabla u_i \cdot \nu &= 0, & x \in \partial\Omega, & \quad t > 0, \\ u_i(x, 0) &= u_{i,0}(x), & x \in \Omega \end{aligned}$$

for all  $i = 1, \dots, N$ , where  $\nu$  is the outward normal on  $\partial\Omega$  and the initial data  $u_{i,0}$  are nonnegative. Here the homogeneous Neumann boundary condition means that the system is closed. The reactions  $f_i(u)$  are derived from all the chemical reactions (1.1) using the *law of mass action*, that is for  $i = 1, \dots, N$ ,

$$(1.3) \quad f_i(u) = \sum_{r=1}^R k_r (y'_{r,i} - y_{r,i}) u^{y_r} \quad \text{with} \quad u^{y_r} = \prod_{i=1}^N u_i^{y_{r,i}}.$$

Note that the nonlinearities are all polynomials. We denote by  $\mu$  the highest order of nonlinearities (1.3), i.e.

$$(1.4) \quad \mu = \max_{y \in \mathcal{C}} |y| \quad \text{with} \quad |y| = \sum_{i=1}^N |y_i| \quad \text{for any } y \in \mathbb{R}^N.$$

It follows straightforwardly there exists  $K > 0$  such that

$$(1.5) \quad |f_i(u)| \leq K(|u|^\mu + 1) \quad \text{for all } i = 1, \dots, N \text{ and all } u \in \mathbb{R}^N.$$

Global existence of solutions is one of the most important questions in studying reaction-diffusion systems (if not any other PDE models). Concerning system (1.2)-(1.3), due to the linear diffusion and polynomial-type nonlinearities, the local existence of solutions follows from standard theory of parabolic systems (see e.g. [1]). The global existence of solutions to (1.2)-(1.3) on the other hand is a subtle task due to the lack of suitable *a priori* estimates (note that the maximum principle fails to apply to reaction-diffusion systems except very special cases). This issue has been extensively addressed in the literature, see e.g. [2, 3, 6, 9, 11, 13, 14, 21, 23] and the survey [20].

Let us briefly mention the recent advances concerning the global existence of (1.2)-(1.3).

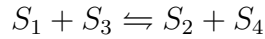
- In [14], by De Giorgi's method (1.2) was proved to have global classical solutions in one and two dimensions with the nonlinearities of order three and two, respectively (which means  $d = 2$  and  $\mu = 1$  or  $d = 2$  and  $\mu = 2$ ). These results have been re-proved (and slightly improved) in [23] using simpler arguments. It was also proved in [3], again by De Giorgi's method, that if the order of nonlinearities is strictly subquadratic (i.e.  $\mu < 2$ ), then classical solutions exist globally in any dimension. See also [13] for quadratic systems in heterostructure.

- In [2] and later in [11] the system (1.2) in higher dimensions and higher order of nonlinearities are shown to possess global classical solutions provided the diffusion coefficients  $d_1, \dots, d_N$  are "close to each other", i.e.

$$\max\{d_i\} - \min\{d_i\} \text{ is small enough.}$$

- Concerning weaker notions of solutions, it was shown in [6, 18] that (1.2) with quadratic nonlinearities has global weak solutions in any dimensions, by using a duality method. An even weaker solution called "renormalized solution" was proved global in [12] for any dimension and any order of nonlinearities.

We remark that despite recent above mentioned advances, the global existence of classical solutions to (1.2) is widely open. For example, it remains unknown whether the "four-species" system modelling



in dimension three (and higher) possesses global classical solutions or not (without assuming the closeness of the diffusion coefficients).

Recently, Cáceres and Cañizo in [4] investigated a different regime of (1.2) so-called close-to-equilibrium regularity. The main idea is that when the solution stays in a small neighbourhood of an equilibrium then it is possible that the linear part is dominating the behaviour of the system, and this helps consequently to obtain more regularity and hence the global existence of classical solutions. The authors proved in [4] that for dimension  $d \leq 4$ , under the assumption that the nonlinearities are at most quadratic and (1.2) satisfies a detailed balanced condition (see Definition 2.1), if initial data is close to an equilibrium in  $L^2$ -norm then the classical solutions exists globally and converge exponentially to the equilibrium in  $L^\infty$ -norm.

The aim of this paper is to extend the results in [4] to systems with complex balance condition and with higher order of nonlinearities.

Our main result reads as follows

**Theorem 1.1.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain with smooth boundary  $\partial\Omega$  (e.g.  $C^{2+\epsilon}$  for  $\epsilon > 0$ ). Assume that the system (1.2) is complex balanced (see Definition 2.1). Assume that  $d \leq 4$  and the order of nonlinearities  $\mu$  (defined in (1.4)) satisfies*

$d = 1$	$d = 2$	$d = 3$	$d = 4$
$\mu = 5$	$\mu = 3$	$\mu = 7/3$	$\mu = 2$

*or equivalently  $\mu = \frac{d+4}{d}$  for  $d \leq 4$ . Finally assume that the initial data  $u_0 = (u_{i,0}) \in L^\infty(\Omega)^N$ .*

*Then there exists  $\epsilon > 0$  such that if the initial data  $u_0$  is close to any strictly positive complex balanced equilibrium  $u_\infty \in (0, \infty)^N$  in  $L^2$ -norm, that is*

$$\sum_{i=1}^N \|u_{i,0} - u_{i,\infty}\|_{L^2(\Omega)} \leq \epsilon,$$

*then the local classical solution to (1.2) exists globally, and converges to  $u_\infty$  in  $L^\infty$ -norm exponentially, i.e.*

$$\sum_{i=1}^N \|u_i(t) - u_{i,\infty}\|_{L^\infty(\Omega)} \leq C e^{-\lambda t} \quad \text{for all } t > 0,$$

with  $C > 0$  and  $\lambda > 0$  are constants.

**Remark 1.2.** *The arguments in the proof of Theorem 1.1 could be extended to higher dimension, i.e.  $d \geq 5$ , under the constraint  $\mu = 1 + 4/d$ . However, this means that the nonlinearities are strictly sub-quadratic and thus the global existence of classical solutions (for any regular enough initial data) was already shown in [3].*

Theorem 1.1 improves the results of [4] in the following senses:

- Firstly, it allows to treat systems with higher order nonlinearities, in particular in one, two and three dimensions. The main idea which leads to this improvement is to utilize the Gagliardo-Nirenberg inequality, especially in one and two dimensions, which helps to treat nonlinearities of order higher than two. Note that the idea was also used in the recent paper [23] to prove the global existence of classical solutions to reaction-diffusion systems in one and two dimensions (without the assumption of initial data being close to equilibrium).
- Secondly, it applies to complex balanced systems which are more general than detailed balanced systems in [4] (see Definition 2.1). One of the main steps in proving Theorem 1.1 is obtaining a spectral gap for the linearised operator. This step was done in [4] thanks to a natural Lyapunov functional of (1.2) inherited from the detailed balanced condition. Extending this to systems with complex balance condition requires nontrivial calculations (see Lemma 3.3).

Last but not least, Theorem 1.1 successfully treats the systems with boundary equilibria, i.e. equilibria belonging to  $\partial\mathbb{R}_+^N$  (see Remark 2.3). It is remarked that for general complex balanced system (1.2) *without boundary equilibria*, it was proved in [5, 10, 17] that any renormalised solution converges exponentially in  $L^1$ -norm to equilibrium. The situation is quite different with the occurrence of boundary equilibria, since in such cases the methods in these forementioned works do not apply. It is however conjectured that for any complex balanced system the strictly positive equilibrium is the only attracting point. This is now named *Global Attractor Conjecture* and has remains as one of the most important open question in chemical reaction networks. Theorem (1.1) shows that despite of boundary equilibria, any solution with initial data being close enough to any strictly positive complex balanced equilibrium in  $L^2$ -norm, converges exponentially to that equilibrium. The results of this paper hence are also the extensions of local stability of complex balanced systems in ODE settings (see [16, 22]).

Let us briefly describe the method used in this paper. First, we consider the linearised system of (1.2) around a strictly positive complex balanced equilibrium. By utilising the complex balanced condition, we obtain a spectral gap for the linearised system. Due to the restrictions on dimensions and order of nonlinearities, it is then shown that the linear part is dominating in the behaviour of (1.2) in the close-to-equilibrium regime, that means the  $L^2$ -norm of solution converges exponentially to the equilibrium. After that, by utilising the Gagliardo-Nirenberg inequality and smoothing effect of the heat operator, we get the global existence of classical solution whose  $L^\infty$ -norm grows at most polynomially in time, which in a combination with the exponential convergence in  $L^2$ -norm leads to the global convergence of solutions to the complex balanced equilibrium in  $L^\infty$ -norm.

The restriction  $d \leq 4$  and corresponding  $\mu = 1 + \frac{4}{d}$  is due to the fact that small  $L^2$ -initial data leads only to control of  $L^2$ -norm of solutions, which is only enough with the current techniques to control the higher norm under the mentioned assumptions on dimension  $d$  and the order of nonlinearity  $\mu$ . The extension to arbitrary dimension seems to need more delicate analysis instead of using only the boundedness in  $L^2$  (see e.g. [19] for a system which has solutions being bounded in  $L^2$  but blowing up in  $L^\infty$ ). For extending this work to higher dimension  $d \geq 5$  and  $\mu \geq 2$ , it is naturally expected that the initial data should be close to equilibrium in  $L^p$ -norm for some  $p > 2$  depending on  $d$  and  $\mu$ . Still the main obstacle is to obtain a global bound in time of  $L^p$ -norm (or  $L^{p+\varepsilon}$ -norm) of the solution. We discuss this extension in more details in Section 5.

**Notations:** Throughout this paper, we denote by  $\|\cdot\|_p$  the norm in  $L^p(\Omega)$  for  $1 \leq p \leq \infty$ . For any other Banach space  $X$  then  $\|\cdot\|_X$  is used for its norm. The inner product in  $L^2(\Omega)$  is written as  $\langle \cdot, \cdot \rangle$ . For any  $T > 0$  we denote by  $Q_T = \Omega \times (0, T)$  and  $L^p(Q_T) = L^p(0, T; L^p(\Omega))$ .

The rest of the paper is organised as follows: In section 2 we recall basic concepts concerning complex balanced chemical reaction networks. Section 3 deals with the linearised system around a strictly positive complex balanced equilibrium and proved that the linearised system has a spectral gap. Section 4 is devoted for the proof of Theorem 1.1. Finally, we discuss in the last section possible extensions to higher dimensions.

## 2. COMPLEX BALANCED CHEMICAL REACTION NETWORKS

In this section, we gather the basic concepts of chemical reaction networks with complex balanced condition. For more details, the interested reader is referred to [7, 8] for the ODE setting and also [5, 10] for the PDE setting. Recalling that we consider  $N$  chemical substances  $S_1, S_2, \dots, S_N$  reacting in  $R$  reactions of the form

$$y_{r,1}S_1 + \dots + y_{r,N}S_N \xrightarrow{k_r} y'_{r,1}S_1 + \dots + y'_{r,N}S_N \quad \text{or shortly} \quad y_r \xrightarrow{k_r} y'_r$$

where  $y_r, y'_r \in (\{0\} \cup [1, \infty))^N$  are stoichiometric coefficients. Under the assumptions of no out-flux and using the law of mass action, one arrives at the reaction-diffusion system (1.2). Since the system under consideration is closed, there are often conservation of masses. Indeed, denote by  $W = (y'_r - y_r)_{r=1, \dots, R} \in \mathbb{R}^{N \times R}$  the Wegscheider's matrix. Let  $m = \text{codim}(W)$ . In case  $m > 0$ , there exists a matrix  $\mathbb{Q} \in \mathbb{R}^{m \times N}$  whose rows are basis of  $\ker(W^\top)$ . Since

$$(2.1) \quad f(u) = (f_1(u), \dots, f_N(u)) = \sum_{r=1}^R k_r (y'_r - y_r) u^{y_r} \in \text{range}(W)$$

we have, thanks to the homogeneous Neumann boundary condition, formally

$$\frac{d}{dt} \int_{\Omega} \mathbb{Q} u dx = \int_{\Omega} \mathbb{Q} f(u) dx = 0,$$

thus

$$(2.2) \quad \mathbb{Q} \bar{u}(t) = \mathbb{Q} \bar{u}_0 =: M \in \mathbb{R}^m \quad \text{for all } t > 0$$

where  $\bar{u} = (\bar{u}_1, \dots, \bar{u}_N)$  with  $\bar{u}_i = \frac{1}{|\Omega|} \int_{\Omega} u_i dx$ , is the vector of averages. In the case  $m = 0$ , then the system (1.2) has no conservation laws.

There are two important classes of homogeneous (chemical) equilibrium for (1.2): *detailed balanced equilibrium* and *complex balanced equilibrium*.

**Definition 2.1.** Let  $u_\infty \in [0, \infty)^N$  be a spatially homogeneous state.

- $u_\infty$  is called a *detailed balanced equilibrium* for (1.2) if for each forward reaction  $y \xrightarrow{k_f} y'$  there exists a corresponding backward reaction  $y' \xrightarrow{k_b} y$  and these two reactions are balanced at  $u_\infty$ , i.e.

$$k_f u_\infty^y = k_b u_\infty^{y'}.$$

- $u_\infty$  is called a *complex balanced equilibrium* for (1.2) if at this equilibrium the total in-flow and out-flow are balanced for each complex, i.e. for any  $y \in \mathcal{C}$  there holds

$$(2.3) \quad \sum_{\{r: y_r=y\}} k_r u_\infty^{y_r} = \sum_{\{r: y'_r=y\}} k_r u_\infty^{y_r}.$$

It is straightforward that

$$u_\infty \text{ is a detailed balanced equilibrium} \implies u_\infty \text{ is a complex balanced equilibrium}$$

but the reverse is in general not true.

If system (1.2) admits a complex balanced equilibrium, then it is called a *complex balanced system*. Note that the condition (2.3) does not give a unique complex balanced equilibrium but a manifold of equilibria instead. The uniqueness of (positive) complex balanced equilibrium is nevertheless determined via the set of conservation laws (2.2).

**Proposition 2.2** (Uniqueness of positive equilibrium). [7, 8] Assume that (1.2) is complex balanced. Then for each initial mass vector  $0 \neq M \in \mathbb{R}^m$  in (2.2), there exists a unique strictly positive complex balanced equilibrium  $u_\infty \in (0, \infty)^N$  to (1.2).

**Remark 2.3.** The strictly positive complex balanced equilibrium is uniquely determined through the initial mass vector  $M$  rather the initial data. That means it's possible to have the same complex balanced equilibrium for different initial data as long as they have the same initial mass.

It is remarked that though (1.2) possesses a unique strictly positive complex balanced equilibrium, it may have possibly many so-called boundary equilibrium  $u_*$ , that means  $u_* \in \partial \mathbb{R}_+^N$  and  $u_*$  satisfies the condition (2.3). For complex balanced systems without boundary equilibria it was proved that any solution converges exponentially to the strictly positive complex balanced equilibrium, see [5, 10]. If a complex balanced system possesses boundary equilibria, then the large time behaviour of solutions is in general unclear. The Global Attractor Conjecture asserts that despite of boundary equilibria, the strictly positive one is still the only attracting point. This conjecture however still remains unsolved in full generality.

The following necessary condition of complex balanced system is needed in Lemma 3.1.

**Proposition 2.4.** Assume that the system (1.2) is complex balanced, then chemical reaction network it is weakly reversible, that is for any reaction  $y_r \rightarrow y'_r$  there exists a chain of reactions starting from  $y'_r$  and finishing at  $y_r$  or more precisely there exist  $y_{k_1}, \dots, y_{k_s} \in \mathcal{C}$  such that the reactions  $y'_r \rightarrow y_{k_1} \rightarrow \dots \rightarrow y_{k_s} \rightarrow y_r$  take place.

The class of complex balanced systems is an important class in chemical reaction network theory and thus has been studied extensively in the last decades. We emphasise that most of the existing works dealt with complex balanced systems in ODE settings, while the PDE settings are much less investigated. We refer the interested reader to recent works [5, 10] for studies of complex balanced systems in PDE settings.

### 3. SPECTRAL GAP FOR THE LINEARISED OPERATOR

This section shows that if (1.2) is complex balanced then its linearised system around any strictly complex balanced equilibrium converges exponentially to that equilibrium. Denote by  $u_\infty$  a strictly complex balanced equilibrium to (1.2), i.e.  $u_\infty \in (0, \infty)^N$  and  $u_\infty$  satisfies (2.3). We first write down the linearised system of (1.2) around  $u_\infty$ . Since  $f_i(u_\infty) = 0$  for all  $i = 1, \dots, N$  we have

$$f_i(u) = \nabla f_i(u_\infty) \cdot (u - u_\infty) + O(|u - u_\infty|^2).$$

Denoting  $v_i = u_i - u_{i,\infty}$  and  $v = (v_1, \dots, v_N)$  we obtain the linearised system around the equilibrium  $u_\infty$  as follow

$$\begin{aligned} (3.1) \quad & \partial_t v_i - d_i \Delta v_i = \nabla f_i(u_\infty) \cdot v =: L_i v, \quad x \in \Omega, \quad t > 0, \\ & \nabla v_i \cdot \nu = 0, \quad x \in \partial\Omega, \quad t > 0, \\ & v_i(x, 0) = u_{i,0}(x) - u_{i,\infty}, \quad x \in \Omega, \end{aligned}$$

with

$$(3.2) \quad L_i v = \nabla f_i(u_\infty) \cdot (u - u_\infty) = \sum_{r=1}^R k_r u_\infty^{y_r} \sum_{j=1}^N (y'_{r,i} - y_{r,i}) y_{r,j} \frac{v_j}{u_{j,\infty}}.$$

**Lemma 3.1.** *Assume that  $u_\infty$  is a (strictly positive) complex balanced equilibrium. Then we have the following identity*

$$(3.3) \quad \sum_{i=1}^N L_i v \frac{v_i}{u_{i,\infty}} = -\frac{1}{2} \sum_{r=1}^R k_r u_\infty^{y_r} \left( \sum_{i=1}^N (y'_{r,i} - y_{r,i}) \frac{v_i}{u_{i,\infty}} \right)^2 \quad \text{for all } v \in \mathbb{R}^N.$$

*Proof.* We proof the identity by comparing the coefficients  $\frac{v_i v_j}{u_{i,\infty} u_{j,\infty}}$  for all  $i, j = 1, \dots, N$ . First of all the left hand side of (3.3) is rewritten as

$$\begin{aligned} (3.4) \quad \sum_{i=1}^N L_i v \frac{v_i}{u_{i,\infty}} &= \sum_{r=1}^R k_r u_\infty^{y_r} \sum_{i=1}^N \sum_{j=1}^N (y'_{r,i} - y_{r,i}) y_{r,j} \frac{v_i}{u_{i,\infty}} \frac{v_j}{u_{j,\infty}} \\ &= \sum_{r=1}^R k_r u_\infty^{y_r} \sum_{i=1}^N \sum_{j=1}^N (y'_{r,i} y_{r,j} - y_{r,i} y_{r,j}) \frac{v_i}{u_{i,\infty}} \frac{v_j}{u_{j,\infty}}. \end{aligned}$$

Hence, for any  $i = 1, \dots, N$ , the coefficient of  $v_i^2 / u_{i,\infty}^2$  in the left hand side of (3.3) is

$$(3.5) \quad \sum_{r=1}^R k_r u_\infty^{y_r} (y'_{r,i} y_{r,i} - y_{r,i}^2)$$



and the coefficient in the right hand side is

$$(3.6) \quad \frac{1}{2} \sum_{r=1}^R k_r u_\infty^{y_r} (2y_{r,i} y'_{r,i} - y_{r,i}^2 - y_{r,i}'^2) = \sum_{r=1}^R k_r u_\infty^{y_r} (y'_{r,i} y_{r,i} - \frac{1}{2}(y_{r,i}^2 + y_{r,i}'^2)).$$

For (3.5) and (3.6) to be equal, we need to show

$$(3.7) \quad \sum_{r=1}^R k_r u_\infty^{y_r} y_{r,i}^2 = \sum_{r=1}^R k_r u_\infty^{y_r} y_{r,i}'^2.$$

Since the system is complex balanced, and thus the corresponding chemical reaction network is weakly reversible (thanks to Proposition 2.4), we have the following identity

$$\sum_{r=1}^R = \sum_{\{y \in \mathcal{C}\}} \sum_{\{r: y_r = y\}} = \sum_{\{y \in \mathcal{C}\}} \sum_{\{r: y'_r = y\}}.$$

Therefore (3.7) is rewritten as

$$\sum_{\{y \in \mathcal{C}\}} \sum_{\{r: y_r = y\}} k_r u_\infty^{y_r} y_{r,i}^2 = \sum_{\{y \in \mathcal{C}\}} \sum_{\{r: y'_r = y\}} k_r u_\infty^{y_r} y_{r,i}'^2,$$

or equivalently

$$\sum_{\{y \in \mathcal{C}\}} y_i^2 \sum_{\{r: y_r = y\}} k_r u_\infty^{y_r} = \sum_{\{y \in \mathcal{C}\}} y_i^2 \sum_{\{r: y'_r = y\}} k_r u_\infty^{y_r}.$$

This equality follows directly from the complex balance condition (2.3) by multiplying both sides of (2.3) with  $y_i^2$  then summing over  $y \in \mathcal{C}$ . Hence, the coefficients of  $v_i^2/u_{i,\infty}^2$  on both sides of (3.3) are equal.

Now for the coefficients of  $v_i v_j / (u_{i,\infty} u_{j,\infty})$  we need to show, from (3.3), that

$$\sum_{r=1}^R k_r u_\infty^{y_r} (y'_{r,i} y_{r,j} - y_{r,i} y_{r,j} + y'_{r,j} y_{r,i} - y_{r,j} y_{r,i}) = - \sum_{r=1}^R k_r u_\infty^{y_r} (y'_{r,i} - y_{r,i})(y'_{r,j} - y_{r,j})$$

or equivalently

$$\sum_{r=1}^R k_r u_\infty^{y_r} y_{r,i} y_{r,j} = \sum_{r=1}^R k_r u_\infty^{y_r} y'_{r,i} y'_{r,j}.$$

This can be proved similarly to (3.7) so we omit it here. ■

**Remark 3.2.** *The proof of Lemma 3.1 has to utilise the complex balanced condition (2.3) and hence is a nontrivial extension from the case of detailed balanced condition systems in [4].*

The next crucial lemma is the main part of this section. It shows that the linearised operator has a spectral gap. This results mostly from the identity (3.3) in Lemma 3.1 and the conservation laws (2.2) of (1.2). Note that we have  $\mathbb{Q} \bar{u}(t) = M = \mathbb{Q} u_\infty$ , thus  $\mathbb{Q} \bar{v}(t) = 0$  for all  $t > 0$ .



**Lemma 3.3.** *There exists a constant  $\lambda > 0$  such that, for any  $v \in H^1(\Omega)^N$  satisfying  $\mathbb{Q}\bar{v} = 0$  and  $|\bar{v}| \in [0, K]^N$  for some  $K > 0$  there holds*

$$-\sum_{i=1}^N d_i \left\langle \Delta v_i, \frac{v_i}{u_{i,\infty}} \right\rangle - \sum_{i=1}^N \left\langle L_i v, \frac{v_i}{u_{i,\infty}} \right\rangle \geq \lambda \sum_{i=1}^N \frac{\|v_i\|_2^2}{u_{i,\infty}}$$

recalling that  $\langle \cdot, \cdot \rangle$  is the inner product in  $L^2(\Omega)$ .

*Proof.* By using Lemma 3.1 and integrations by part we have

$$\begin{aligned} & -\sum_{i=1}^N d_i \left\langle \Delta v_i, \frac{v_i}{u_{i,\infty}} \right\rangle - \sum_{i=1}^N \left\langle L_i v, \frac{v_i}{u_{i,\infty}} \right\rangle \\ &= \sum_{i=1}^N d_i \frac{\|\nabla v_i\|_2^2}{u_{i,\infty}} + \frac{1}{2} \sum_{r=1}^R k_r u_\infty^{y_r} \int_{\Omega} \left( \sum_{i=1}^N (y'_{r,i} - y_{r,i}) \frac{v_i}{u_{i,\infty}} \right)^2 dx. \end{aligned}$$

Firstly, by the Poincaré inequality  $\|\nabla f\|_2^2 \geq P(\Omega) \|f - \bar{f}\|_2^2$ , recalling that  $\bar{f} = \frac{1}{|\Omega|} \int_{\Omega} f dx$  we have

$$(3.8) \quad \sum_{i=1}^N d_i \frac{\|\nabla v_i\|_2^2}{u_{i,\infty}} \geq P(\Omega) \min_{i=1,\dots,N} \{d_i\} \sum_{i=1}^N \frac{\|v_i - \bar{v}_i\|_2^2}{u_{i,\infty}}.$$

On the other hand, by Jensen's inequality we get,

$$\frac{1}{2} \sum_{r=1}^R k_r u_\infty^{y_r} \int_{\Omega} \left( \sum_{i=1}^N (y'_{r,i} - y_{r,i}) \frac{v_i}{u_{i,\infty}} \right)^2 dx \geq \frac{1}{2|\Omega|} \sum_{r=1}^R k_r u_\infty^{y_r} \left( \sum_{i=1}^N (y'_{r,i} - y_{r,i}) \frac{\bar{v}_i}{u_{i,\infty}} \right)^2.$$

We will now prove that there exists  $\delta > 0$  such that for all  $\bar{v} \in \mathbb{R}^N$  with  $\mathbb{Q}\bar{v} = 0$  there holds

$$(3.9) \quad \frac{1}{2|\Omega|} \sum_{r=1}^R k_r u_\infty^{y_r} \left( \sum_{i=1}^N (y'_{r,i} - y_{r,i}) \frac{\bar{v}_i}{u_{i,\infty}} \right)^2 \geq \delta \sum_{i=1}^N \frac{\bar{v}_i^2}{u_{i,\infty}}.$$

We prove (3.9) by a contradiction argument. Assume that

$$(3.10) \quad \inf_{\bar{v} \in \mathbb{R}^N, \mathbb{Q}\bar{v}=0} \frac{\frac{1}{2|\Omega|} \sum_{r=1}^R k_r u_\infty^{y_r} \left( \sum_{i=1}^N (y'_{r,i} - y_{r,i}) \frac{\bar{v}_i}{u_{i,\infty}} \right)^2}{\sum_{i=1}^N \frac{\bar{v}_i^2}{u_{i,\infty}}} = 0.$$

Since both the nominator and denominator have the homogeneity of order two, we can assume w.l.o.g. that  $\|\bar{v}\| = 1$  with  $\|\cdot\|$  denotes the Euclidean norm in  $\mathbb{R}^N$ . Moreover, because  $|\bar{v}| \in [0, K]^N$ , the denominator is bounded from above. Then (3.10) holds only if

$$\sum_{r=1}^R k_r u_\infty^{y_r} \left( \sum_{i=1}^N (y'_{r,i} - y_{r,i}) \frac{\bar{v}_i}{u_{i,\infty}} \right)^2 = 0$$

which means for all  $r = 1, \dots, R$ ,

$$\sum_{i=1}^N (y'_{r,i} - y_{r,i}) \frac{\bar{v}_i}{u_{i,\infty}} = 0.$$

Recalling the Wegscheider's matrix  $W = (y'_r - y_r)_{r=1,\dots,R}$ , then this implies that

$$\text{diag}\left(\frac{1}{u_{1,\infty}}, \dots, \frac{1}{u_{N,\infty}}\right) \bar{v} = \left(\frac{\bar{v}_1}{u_{1,\infty}}, \dots, \frac{\bar{v}_N}{u_{N,\infty}}\right) \in \ker(W^\top).$$

If  $\ker W^\top = \{0\}$  then it follows immediately  $\bar{v} = 0$ , contradicting with  $\|\bar{v}\| = 1$ . In case  $\ker W^\top \neq \{0\}$ , we recall that the rows of  $\mathbb{Q}$  form a basis of  $\ker(W^\top)$  (see Section 2), hence

$$\text{diag}\left(\frac{1}{u_{1,\infty}}, \dots, \frac{1}{u_{N,\infty}}\right) \bar{v} = \mathbb{Q}^\top \xi \quad \text{for some } \xi \in \mathbb{R}^m$$

and thus  $\bar{v} = \text{diag}(u_{1,\infty}, \dots, u_{N,\infty}) \mathbb{Q}^\top \xi$ . By the assumption  $\mathbb{Q} \bar{v} = 0$  we have

$$\mathbb{Q} \text{diag}(u_{1,\infty}, \dots, u_{N,\infty}) \mathbb{Q}^\top \xi = 0$$

which implies  $\xi = 0$  since  $\mathbb{Q}$  has full rank. But  $\xi = 0$  leads to  $\bar{v} = 0$  which contradicts with  $\|\bar{v}\| = 1$ . Therefore (3.9) is proved. Finally, a combination of (3.8) and (3.9) gives us the desired estimate of this Lemma.  $\blacksquare$

**Remark 3.4.** Lemma 3.3 is stated for general functions  $v \in H^1(\Omega)^N$  and thus has to impose the boundedness  $\bar{v} \in [0, K]^N$  for some  $K > 0$ . We remark that this is not an additional assumption when one considers solutions to a complex balanced system (1.2). Indeed, it is well known (see e.g. [5]) that if (1.2) is complex balanced then the free energy (or relative entropy)

$$\mathcal{E}(u|u_\infty) = \int_{\Omega} \left( u_i \log \frac{u_i}{u_{i,\infty}} - u_i + u_{i,\infty} \right) dx$$

is decreasing along the trajectory. Hence for all  $t > 0$

$$\mathcal{E}(u(t)|u_\infty) \leq \mathcal{E}(u_0|u_\infty).$$

We then obtain from this by using the elementary inequalities  $x \log(x/y) - x + y \geq (\sqrt{x} - \sqrt{y})^2 \geq \frac{1}{2}x - y$  that

$$\sum_{i=1}^N \bar{u}_i(t) \leq 2 \left( \sum_{i=1}^N u_{i,\infty} + \mathcal{E}(u_0|u_\infty) \right)$$

This combining with the nonnegativity of  $u_i$  leads to the boundedness of  $\bar{u}$  and thus for the difference  $\bar{v} = \bar{u} - u_\infty$ .

#### 4. PROOF OF THEOREM 1.1

In this section we denote again by  $u_\infty$  a strictly positive complex balanced equilibrium to (1.2). By using Taylor's expansion of the nonlinearities around the equilibrium  $u_\infty$ , system (1.2) is rewritten as

$$\begin{aligned} \partial_t u_i - d_i \Delta u_i &= \nabla f_i(u_\infty) \cdot (u - u_\infty) + \frac{1}{2} (u - u_\infty)^\top \nabla^2 f_i(\xi) (u - u_\infty), & x \in \Omega, \\ \nabla u_i \cdot \nu &= 0, & x \in \partial\Omega, \\ u_i(x, 0) &= u_{i,0}(x), & x \in \Omega \end{aligned}$$

where  $\xi = \theta u + (1 - \theta)u_\infty = u_\infty + \theta(u - u_\infty)$  for some  $\theta \in (0, 1)$ . Denote by  $v_i = u_i - u_{i,\infty}$ , the system for  $v = (v_1, \dots, v_N)$  reads as

$$(4.1) \quad \begin{aligned} \partial_t v_i - d_i \Delta v_i &= L_i v + \frac{1}{2} v^\top \nabla^2 f_i(\xi) v, & x \in \Omega, \\ \nabla v_i \cdot \nu &= 0, & x \in \partial\Omega, \\ v_i(x, 0) &= u_{i,0}(x) - u_{i,\infty}, & x \in \Omega \end{aligned}$$

recalling that  $L_i v$  is defined in (3.2). From the growth condition (1.5)  $|f_i(u)| \leq K(|u|^\mu + 1)$  and the fact that  $f(u)$  is a polynomial, we get

$$|\nabla^2 f(\xi)| \leq C(|\xi|^{\mu-2} + 1) \leq C(|v|^{\mu-2} + 1)$$

since  $\xi = u_\infty + \theta v$ , and consequently

$$(4.2) \quad |v^\top \nabla^2 f_i(\xi) v| \leq C(|v|^\mu + |v|^2) \leq C \sum_{j=1}^N (|v_j|^\mu + |v_j|^2).$$

**Lemma 4.1.** *There exists  $\varepsilon > 0$  small enough such that if  $\sum_{i=1}^N \|v_{i,0}\|_2 \leq \varepsilon$  then the solution  $v = (v_1, \dots, v_N)$  to (4.1) decays exponentially to zero in  $L^2$ -norm, or more precisely*

$$\sum_{i=1}^N \frac{\|v_i(t)\|_2^2}{u_{i,\infty}} \leq e^{-\delta t} \sum_{i=1}^N \frac{\|v_{i,0}\|_2^2}{u_{i,\infty}} \quad \text{for all } t > 0,$$

where  $\delta > 0$  is a constant independent of  $t > 0$ .

*Proof.* Multiplying (4.1) with  $v_i/u_{i,\infty}$  and summing over  $i = 1, \dots, N$ , it yields with the help of (4.2), Lemma 3.3 and Hölder's inequality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \sum_{i=1}^N \frac{\|v_i\|_2^2}{u_{i,\infty}} + \frac{1}{2} \sum_{i=1}^N d_i \frac{\|\nabla v_i\|_2^2}{u_{i,\infty}} + \frac{\lambda}{2} \sum_{i=1}^N \frac{\|v_i\|_2^2}{u_{i,\infty}} &\leq \frac{1}{2} \int_\Omega \sum_{i=1}^N \left| \frac{v_i}{u_{i,\infty}} \right| |v^\top \nabla^2 f_i(\xi) v| dx \\ &\leq C \int_\Omega \sum_{i=1}^N \left| \frac{v_i}{u_{i,\infty}} \right| \sum_{j=1}^N (|v_j|^\mu + |v_j|^2) dx \\ &\leq C \sum_{i=1}^N (\|v_i\|_{\mu+1}^{\mu+1} + \|v_i\|_3^3), \end{aligned}$$

thus for some  $\delta > 0$ ,

$$(4.3) \quad \frac{1}{2} \frac{d}{dt} \sum_{i=1}^N \frac{\|v_i\|_2^2}{u_{i,\infty}} + \frac{\delta}{2} \sum_{i=1}^N \frac{\|v_i\|_2^2}{u_{i,\infty}} + \frac{\delta}{2} \sum_{i=1}^N \|v_i\|_{H^1(\Omega)}^2 \leq C \sum_{i=1}^N (\|v_i\|_{\mu+1}^{\mu+1} + \|v_i\|_3^3).$$

Since  $d \leq 4$ , the Sobolev embedding yields  $H^1(\Omega) \hookrightarrow L^4(\Omega)$ . By using the interpolation inequality we get

$$\|v_i\|_3^3 \leq \|v_i\|_4^2 \|v_i\|_2 \leq C \|v_i\|_{H^1(\Omega)}^2 \|v_i\|_2.$$

For the term  $\|v_i\|_{\mu+1}^{\mu+1}$  we apply the Gagliardo-Nirenberg inequality to obtain, recalling that  $\mu = \frac{d+4}{d}$ ,

$$\|v_i\|_{\mu+1}^{\mu+1} = \|v_i\|_{2(d+2)/d}^{2(d+2)/d} \leq C \|v_i\|_{H^1(\Omega)}^2 \|v_i\|_2^{4/d}.$$

Therefore, it follows from (4.3) that

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^N \frac{\|v_i\|_2^2}{u_{i,\infty}} + \frac{\delta}{2} \sum_{i=1}^N \frac{\|v_i\|_2^2}{u_{i,\infty}} + \frac{\delta}{2} \sum_{i=1}^N \|v_i\|_{H^1(\Omega)}^2 \leq C \sum_{i=1}^N \|v_i\|_{H^1(\Omega)}^2 (\|v_i\|_2 + \|v_i\|_2^{4/d}).$$

Hence, with the assumption  $\sum_{i=1}^N \|v_{i,0}\|_2 \leq \varepsilon$  with  $\varepsilon$  small enough we get

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^N \frac{\|v_i\|_2^2}{u_{i,\infty}} + \frac{\delta}{2} \sum_{i=1}^N \frac{\|v_i\|_2^2}{u_{i,\infty}} \leq 0$$

and consequently, by a Gronwall's lemma,

$$\sum_{i=1}^N \frac{\|v_i(t)\|_2^2}{u_{i,\infty}} \leq e^{-\delta t} \sum_{i=1}^N \frac{\|v_{i,0}\|_2^2}{u_{i,\infty}}.$$

■

**Important notation:** From now on we always denote by  $C_T$  a positive constant which depends *at most polynomially* on  $T$ , that is there exists a polynomial  $P(x)$  such that  $C_T \leq P(T)$  for all  $T > 0$ .

The following two lemmas are important in getting global classical solutions to (4.1).

**Lemma 4.2.** [2, Lemma 3.3] *Let  $f \in L^p(Q_T)$  with  $p \geq 1$  and  $u$  is the solution to the heat equation*

$$(4.4) \quad \begin{aligned} y_t - d\Delta y &= f, & x \in \Omega, \\ \nabla y \cdot \nu &= 0, & x \in \partial\Omega, \\ y(x, 0) &= y_0(x), & x \in \Omega \end{aligned}$$

with  $y_0 \in L^\infty(\Omega)$ .

- If  $p < (d+2)/2$  then

$$(4.5) \quad \|y\|_{L^s(Q_T)} \leq C_T \quad \text{for all} \quad s < \frac{(d+2)p}{d+2-2p}.$$

- If  $p \geq (d+2)/2$  then

$$(4.6) \quad \|y\|_{L^r(Q_T)} \leq C_T \quad \text{for all} \quad 1 \leq r < +\infty.$$

**Lemma 4.3.** [4] *Consider the heat equation (4.4). If  $f \in L^\infty(0, T; L^p(\Omega))$  with  $p > d/2$  then*

$$(4.7) \quad \|y\|_{L^\infty(Q_T)} \leq C_T$$

with  $C_T$  is a constant depending at most polynomially on  $T$ .

**Remark 4.4.** We remark that both regularity results in Lemmas 4.2 and 4.3 are classical and can be found in the literature, for instance in the classical book [15]. Here we emphasise in (4.5), (4.6) and (4.7) the polynomial dependence of the constant  $C_T$  on  $T$  since it's important in proving Theorem 1.1. In fact, the exponential decay in  $L^\infty$ -norm is a result of the exponential decay in lower norm, say  $L^2$ -norm in Lemma 4.1, and the polynomial growth in  $T$  of higher order norms (see e.g. (4.12)).

We are now ready to give

*Proof of Theorem 1.1.* Multiplying (4.1) with  $p|v_i|^{p-2}v_i$  (or more precisely a smoothed approximation of  $p|v_i|^{p-2}v_i$  and letting the smoothing go to zero) then integrating over  $\Omega$ , we have

$$(4.8) \quad \frac{d}{dt} \|v_i\|_p^p + d_i p(p-1) \int_{\Omega} |v_i|^{p-2} |\nabla v_i|^2 dx = p \int_{\Omega} \left( L_i v + \frac{1}{2} v^\top \nabla^2 f_i(\xi) v \right) |v_i|^{p-2} v_i dx.$$

For the second term we have

$$d_i p(p-1) \int_{\Omega} |v_i|^{p-2} |\nabla v_i|^2 dx = \frac{4d_i(p-1)}{p} \|\nabla(v_i^{p/2})\|_2^2.$$

For the right hand side of (4.8), since  $L_i v$  is linear (see (3.2)) we estimate by Hölder's inequality

$$\int_{\Omega} L_i v |v_i|^{p-2} v_i dx \leq \int_{\Omega} |L_i v| |v_i|^{p-1} dx \leq C \sum_{i=1}^N \|v_i\|_p^p.$$

For the term concerning the nonlinearities we use (4.2) and Hölder's inequality to obtain

$$\int_{\Omega} v^\top \nabla^2 f_i(\xi) v |v_i|^{p-2} v_i dx \leq C \int_{\Omega} |v_i|^{p-1} \sum_{j=1}^N (|v_j|^\mu + |v_j|^2) dx \leq C \sum_{i=1}^N (\|v_i\|_{\mu+p-1}^{\mu+p-1} + \|v_i\|_{p+1}^{p+1}).$$

Therefore we obtain from summing (4.8) over  $i = 1, \dots, N$  that

$$(4.9) \quad \frac{d}{dt} \sum_{i=1}^N \|v_i\|_p^p + C \sum_{i=1}^N \|\nabla(v_i^{p/2})\|_2^2 \leq C \sum_{i=1}^N (\|v_i\|_{\mu+p-1}^{\mu+p-1} + \|v_i\|_{p+1}^{p+1} + \|v_i\|_p^p).$$

Since  $\Omega$  is bounded and  $\mu \geq 2$  we have

$$\|v_i\|_p^p \leq \|v_i\|_{\mu+p-1}^p \leq C(\|v_i\|_{\mu+p-1}^{\mu+p-1} + 1) \quad \text{and} \quad \|v_i\|_{p+1}^{p+1} \leq C(\|v_i\|_{\mu+p-1}^{\mu+p-1} + 1).$$

Then by adding  $\sum_{i=1}^N \|v_i\|_p^p = \sum_{i=1}^N \|v_i^{p/2}\|_2^2$  to both sides of (4.9) we obtain

$$(4.10) \quad \frac{d}{dt} \sum_{i=1}^N \|v_i\|_p^p + C \sum_{i=1}^N \|v_i^{p/2}\|_{H^1(\Omega)}^2 \leq C \sum_{i=1}^N \|v_i\|_{\mu+p-1}^{\mu+p-1} + C.$$

We now consider the cases  $d = 1$ ,  $d = 2$ ,  $d = 3$  and  $d = 4$  separately due to the different Sobolev embeddings in each case.

**The case  $d = 1$ ,** which implies  $\mu = 5$ . Choosing  $p = 4$  in (4.10) we have

$$\frac{d}{dt} \sum_{i=1}^N \|v_i\|_4^4 + C \sum_{i=1}^N \|v_i^2\|_{H^1(\Omega)}^2 \leq C \sum_{i=1}^N \|v_i\|_8^8 + C = C \sum_{i=1}^N \|v_i^2\|_4^4 + C.$$

Applying the Gagliardo-Nirenberg inequality in one dimension

$$\|v_i^2\|_4^4 \leq C \|v_i^2\|_{H^1(\Omega)}^2 \|v_i^2\|_1^2 = C \|v_i^2\|_{H^1(\Omega)}^2 \|v_i\|_2^4.$$

Thus

$$(4.11) \quad \frac{d}{dt} \sum_{i=1}^N \|v_i\|_4^4 + C \sum_{i=1}^4 \|v_i^2\|_{H^1(\Omega)}^2 \leq C \sum_{i=1}^N \|v_i^2\|_{H^1(\Omega)}^2 (\|v_i\|_2^2 - 1) + C.$$

Thanks to Lemma 4.1, when  $\sum_{i=1}^N \|v_{i,0}\|_2 \leq \varepsilon$  we have also  $\sum_{i=1}^N \|v_i(t)\|_2 \leq \varepsilon$  for all  $t > 0$ , and hence it follows from (4.11) and the one dimensional embedding  $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$  that

$$\frac{d}{dt} \sum_{i=1}^N \|v_i\|_4^4 + C \sum_{i=1}^N \|v_i\|_\infty^4 \leq C.$$

Integrating on  $(0, T)$  yields

$$\sum_{i=1}^N \|v_i(T)\|_4^4 + C \sum_{i=1}^N \int_0^T \|v_i(\tau)\|_\infty^4 d\tau \leq \sum_{i=1}^N \|v_{i,0}\|_4^4 + CT.$$

That means for all  $i = 1, \dots, N$  it holds  $v_i \in L^\infty(0, T; L^4(\Omega)) \cap L^4(0, T; L^\infty(\Omega))$ . It then follows from interpolation that  $v_i \in L^8(Q_T)$  for all  $i = 1, \dots, N$ . This implies that the right hand side of the equation in (4.1) belongs to  $L^{8/5}(Q_T)$ , which with the help of Lemma 4.2(ii) leads to  $v_i \in L^r(Q_T)$  for all  $r \in [1, \infty)$  and  $\|v_i\|_{L^r(Q_T)} \leq C_T$ . Hence we obtain  $\|v_i(T)\|_{H^2(\Omega)} \leq C_T$  and by interpolation

$$(4.12) \quad \|v_i(T)\|_\infty \leq C \|v_i(T)\|_{H^2(\Omega)}^\theta \|v_i(T)\|_2^{1-\theta} \leq C_T e^{-\delta(1-\theta)T} \leq C e^{-\delta'T}$$

for some  $0 < \delta' < \delta(1 - \theta)$ , which completes the proof of Theorem in one dimension.

**The case  $d = 2$  with  $\mu = 3$ .** Choosing again  $p = 4$  in (4.10) and applying the two dimensional Gagliardo-Nirenberg inequality

$$\|v_i\|_6^6 = \|v_i^2\|_3^3 \leq C \|v_i^2\|_{H^1(\Omega)}^2 \|v_i^2\|_1 = C \|v_i^2\|_{H^1(\Omega)}^2 \|v_i\|_2^2$$

we get from (4.10) that

$$\frac{d}{dt} \sum_{i=1}^N \|v_i\|_4^4 + C \sum_{i=1}^N \|v_i^2\|_{H^1(\Omega)}^2 \leq C \sum_{i=1}^N \|v_i^2\|_{H^1(\Omega)}^2 \|v_i\|_2^2 + C.$$

By using  $\sum_{i=1}^N \|v_i\|_2$  is small enough, and the two dimensional embedding  $H^1(\Omega) \hookrightarrow L^q(\Omega)$  for any  $q \in [1, \infty)$  we obtain

$$\frac{d}{dt} \sum_{i=1}^N \|v_i\|_4^4 + C \sum_{i=1}^N \|v_i\|_{2q}^4 \leq C$$

and consequently, by integrating on  $(0, T)$ ,

$$\sum_{i=1}^N \|v_i(T)\|_4^4 + C \sum_{i=1}^N \int_0^T \|v_i(\tau)\|_{2q}^4 d\tau \leq \sum_{i=1}^N \|v_{i,0}\|_4^4 + CT.$$

Therefore  $v_i \in L^\infty(0, T; L^4(\Omega)) \cap L^4(0, T; L^{2q}(\Omega)) \hookrightarrow L^{8-8/q}(Q_T)$  by interpolation. Since  $q \in [1, \infty)$  arbitrary, it holds in fact  $v_i \in L^r(Q_T)$  for all  $1 \leq r < 8$ . Because  $\mu = 3$ , the right hand side of (4.1) belongs to  $L^s(Q_T)$  for all  $1 \leq s < 8/3$ , and in particular in  $L^{2+\alpha}(Q_T)$  with some  $\alpha > 0$ . Lemma 4.2(ii) implies  $v_i \in L^r(Q_T)$  for all  $1 \leq r < +\infty$ . An interpolation argument similar to (4.12) concludes the proof of Theorem 1.1 in two dimensions.

**The case  $d = 3$  and  $\mu = 7/3$ .** In this case we choose  $p > 2$  arbitrary. Using the dimensional Sobolev embedding  $H^1(\Omega) \hookrightarrow L^6(\Omega)$  in (4.10) we have

$$\frac{d}{dt} \sum_{i=1}^N \|v_i\|_p^p + C \sum_{i=1}^N \|v_i\|_{3p}^p \leq C \sum_{i=1}^N \|v_i\|_{p+4/3}^{p+4/3} + C.$$

By the interpolation inequality

$$\|v_i\|_{p+4/3}^{p+4/3} \leq \|v_i\|_{3p}^p \|v_i\|_2^{4/3}$$

and  $\sum_{i=1}^N \|v_i\|_2$  is small, we arrive at

$$\frac{d}{dt} \sum_{i=1}^N \|v_i\|_p^p \leq C$$

and consequently

$$\sum_{i=1}^N \|v_i(T)\|_p^p \leq \sum_{i=1}^N \|v_{i,0}\|_p^p + CT,$$

which means  $v_i \in L^\infty(0, T; L^p(\Omega))$  for any  $p \in [1, \infty)$ , and hence the right hand side of (4.10) belongs to  $L^\infty(0, T; L^p(\Omega))$  for any  $p \in [1, \infty)$ . This is enough to apply Lemma 4.3 to obtain  $v_i \in L^\infty(Q_T)$  with  $\|v_i\|_{L^\infty(Q_T)} \leq C_T$  and thus concludes Theorem 1.1 in three dimensions thanks to an argument similarly to (4.12).

**The case  $d = 4$  and  $\mu = 2$ .** This case is similar to the case when  $d = 3$ , and in fact was proved in [4] for systems satisfying detail balance condition. For  $p > 2$ , it follows from the Sobolev embedding in four dimensions  $H^1(\Omega) \hookrightarrow L^4(\Omega)$  and (4.10) that

$$\frac{d}{dt} \sum_{i=1}^N \|v_i\|_p^p + C \sum_{i=1}^N \|v_i\|_{2p}^p \leq C \sum_{i=1}^N \|v_i\|_{p+1}^{p+1} + C \leq C \sum_{i=1}^N \|v_i\|_{2p}^p \|v_i\|_2 + C$$

in which the last inequality follows from interpolation inequality. The rest follows exactly as the case  $d = 3$  so we omit it here. ■

## 5. EXTENSION TO HIGHER DIMENSIONS

As mentioned in the introduction, the main advantage of the  $L^2$ -setting is the spectral gap of the linearised operator which leads to a global bound on  $L^2$ -norm of solutions. This however also leads to the restriction  $d \leq 4$  and  $\mu = 1 + \frac{4}{d}$  due to the fact that the bound of  $L^2$ -norm is in general not enough to control the nonlinearities of higher order and in higher dimensions. It is natural to expect that in such cases the initial data need to be small in  $L^p$ -norm, with some  $p > 2$  depending on  $d$  and  $\mu$ . More importantly, one needs to prove the control of solution in  $L^p$ -norm (or  $L^{p+\varepsilon}$ -norm) to obtain the regularity of solution up to  $L^\infty$ . This remains as an open question. In this section we show how to proceed once the suitable control on solutions is established.

**Proposition 5.1.** *Let  $d \geq 3$ . Consider the linearised system (4.1). Let*

$$(5.1) \quad p_0 = \frac{d(\mu - 1)}{2}.$$



Assume either

$$(5.2) \quad \sum_{i=1}^N \|v_i(t)\|_{p_0} \leq \varepsilon \quad \text{for all } t > 0,$$

for small  $\varepsilon$  or

$$(5.3) \quad \sum_{i=1}^N \|v_i(t)\|_{p_0+\delta} \leq \widehat{C}(t) \quad \text{for all } t > 0,$$

for some positive constant  $\delta > 0$  and  $\widehat{C}(t)$  is a continuous function. Then the classical solution  $v$  to (4.1), and consequently the classical solution  $u$  to (1.2), exists globally.

**Remark 5.2.** In the case, for instance,  $d = 4$  and  $\mu = 2$ , which implies  $p_0 = 2$ , the assumption (5.2) in fact can be proved thanks to a spectral gap of the linearised operator (see Lemma 4.1).

*Proof.* Let  $p \geq p_0$  arbitrary. Multiplying (4.1) by  $p|v_i|^{p-2}v_i$  and proceed as the proof of Theorem 1.1 we obtain (4.10), which we rewrite here

$$\frac{d}{dt} \sum_{i=1}^N \|v_i\|_p^p + C \sum_{i=1}^N \|v_i^{p/2}\|_{H^1(\Omega)}^2 \leq C \sum_{i=1}^N \|v_i\|_{\mu+p-1}^{\mu+p-1} + C.$$

By the Sobolev embedding  $H^1(\Omega) \hookrightarrow L^{2d/(d-2)}(\Omega)$ , recalling that  $d \geq 3$ , we have

$$\|v_i^{p/2}\|_{H^1(\Omega)}^2 \geq C \|v_i^{p/2}\|_{2d/(d-2)}^2 = C \|v_i\|_{dp/(d-2)}^p =: C \|v_i\|_q^p$$

with  $q = \frac{dp}{p-2}$ . Therefore we obtain

$$(5.4) \quad \frac{d}{dt} \sum_{i=1}^N \|v_i\|_p^p + C \sum_{i=1}^N \|v_i\|_q^p \leq C \sum_{i=1}^N \|v_i\|_{\mu+p-1}^{\mu+p-1} + C.$$

Note that  $p \geq p_0 = \frac{d(\mu-1)}{2}$  implies  $q = \frac{dp}{d-2} \geq \mu + p - 1$  and  $\mu + p - 1 \geq p_0$  since  $\mu \geq 1$ .

We consider first the case when the assumption (5.2) is satisfied, we use the interpolation inequality to have, recalling that  $p_0 \leq \mu + p - 1 \leq q$ ,

$$(5.5) \quad \|v_i\|_{\mu+p-1}^{\mu+p-1} \leq \|v_i\|_q^{\theta(\mu+p-1)} \|v_i\|_{p_0}^{(1-\theta)(\mu+p-1)}$$

with  $\theta \in (0, 1)$  satisfies

$$\frac{1}{\mu + p - 1} = \frac{\theta}{q} + \frac{1 - \theta}{p_0} \implies \theta = \frac{q[(\mu + p - 1) - p_0]}{(q - p_0)(\mu + p - 1)}.$$

From  $q = dp/(d-2)$  and  $p_0 = \frac{d(\mu-1)}{2}$  we can easily verify that

$$\theta(\mu + p - 1) = \frac{q[(\mu + p - 1) - p_0]}{q - p_0} = p.$$

Therefore

$$\|v_i\|_{\mu+p-1}^{\mu+p-1} \leq \|v_i\|_q^p \|v_i\|_{p_0}^{(1-\theta)(\mu+p-1)}$$

Inserting this into (5.4) we obtain

$$\frac{d}{dt} \sum_{i=1}^N \|v_i\|_p^p + C \sum_{i=1}^N \|v_i\|_q^p \leq C \sum_{i=1}^N \|v_i\|_q^p \|v_i\|_{p_0}^{(1-\theta)(\mu+p-1)} + C.$$

Thanks to (5.2), with  $\varepsilon$  is small enough we obtain

$$\frac{d}{dt} \sum_{i=1}^N \|v_i\|_p^p \leq C$$

or consequently

$$\sum_{i=1}^N \|v_i(T)\|_p^p \leq \sum_{i=1}^N \|v_{i,0}\|_p^p + CT$$

for arbitrary  $p \geq p_0$ . By applying Lemma 4.3 we obtain the global existence of classical solutions.

In the case when (5.3) holds, we will modify the interpolation inequality (5.5) as follow

$$\|v_i\|_{\mu+p-1}^{\mu+p-1} \leq \|v_i\|_q^{\theta(\mu+p-1)} \|v_i\|_{p_0+\delta}^{(1-\theta)(\mu+p-1)}$$

with  $\theta \in (0, 1)$  solving

$$\frac{1}{\mu+p-1} = \frac{\theta}{q} + \frac{1-\theta}{p_0+\delta} \implies \theta = \frac{q[(\mu+p-1) - p_0 - \delta]}{(q - p_0 - \delta)(\mu+p-1)}.$$

Using again  $q = dp/(d-2)$  and  $p_0 = d(\mu-1)/2$  we can check that

$$\theta(\mu+p-1) = \frac{q[(\mu+p-1) - p_0 - \delta]}{q - p_0 - \delta} < p.$$

Therefore, an application of Young's inequality leads to

$$\|v_i\|_{\mu+p-1}^{\mu+p-1} \leq \|v_i\|_q^{\theta(\mu+p-1)} \|v_i\|_{p_0+\delta}^{(1-\theta)(\mu+p-1)} \leq \kappa \|v_i\|_q^p + C_\kappa \|v_i\|_{p_0+\delta}^{(1-\theta)[\theta(\mu+p-1)-p]/\theta}$$

in which  $\kappa > 0$  can be chosen as small as possible. Inserting this into (5.4) and choosing  $\kappa$  small enough we arrive at

$$\frac{d}{dt} \sum_{i=1}^N \|v_i\|_p^p \leq C \sum_{i=1}^N \|v_i\|_{p_0+\delta}^{(1-\theta)[\theta(\mu+p-1)-p]/\theta} + C \leq C \widehat{C}(t)^{(1-\theta)[\theta(\mu+p-1)-p]/\theta} + C$$

due to the assumption (5.3). While  $\widehat{C}(t)$  is continuous we obtain finally

$$\sum_{i=1}^N \|v_i(T)\|_p^p \leq \sum_{i=1}^N \|v_{i,0}\|_p^p + C \int_0^T \widehat{C}(t)^{(1-\theta)[\theta(\mu+p-1)-p]/\theta} dt + CT < +\infty,$$

for arbitrary  $p \geq p_0 + \delta$ , which together with Lemma 4.3 implies the global existence of classical solutions. The proof of Proposition 5.1 is thus complete.  $\blacksquare$

Note that Proposition 5.1 provides only the global existence of classical solutions but not the uniform-in-time bound as in Theorem 1.1. The reason is that, in contrast to the case of small  $L^2$ -initial data, one does not have in general the convergence of solution  $u$  to the complex balanced equilibrium  $u_\infty$ , or equivalently the decay of  $v = u - u_\infty$  to zero, except the case when the considered system is complex balanced and has no boundary equilibria,

then the solutions are proved to converge to  $u_\infty$  in  $L^1$ -norm without the assumption of initial data being close to  $u_\infty$  (see [5, 10]).

**Corollary 5.3.** *Assume that the system (1.2) is complex balanced and has no boundary equilibria. Suppose moreover that all the assumptions in Proposition 5.1 are satisfied with the function  $\widehat{C}(t)$  in (5.3) grows at most polynomially. Then the classical solution to (1.2) exists globally and converges to the complex balanced equilibrium in  $L^\infty$ -norm.*

*Proof.* By Proposition 5.1 and the assumption that  $\widehat{C}(t)$  grows at most polynomially, we obtain for the classical solution to (1.2) that for all  $i = 1, \dots, N$

$$\|u_i(T)\|_\infty \leq C_T$$

and consequently

$$\|u_i(T)\|_{H^1(\Omega)} \leq C_T$$

where  $C_T$  is a constant grows at most polynomially in  $T$ . Since the system is assumed to have no boundary equilibria, it was proved in [5, 10] that

$$\sum_{i=1}^N \|u_i(t) - u_{i,\infty}\|_1 \leq C e^{-\lambda t} \quad \text{for all } t > 0.$$

Therefore an application of the Gagliardo-Nirenberg inequality concludes the Corollary. ■

**Acknowledgements.** This work is partially supported by International Training Program IGDK 1754 and NAWI Graz.

## REFERENCES

- [1] H. Amann. "Global existence for semilinear parabolic systems." J. reine angew. Math 360 (1985): 47-83.
- [2] J.A. Cañizo, L. Desvillettes and K. Fellner. "Improved duality estimates and applications to reaction-diffusion equations." Communications in Partial Differential Equations 39.6 (2014): 1185-1204.
- [3] M.C. Caputo and A. Vasseur. "Global regularity of solutions to systems of reaction diffusion with sub-quadratic growth in any dimension." Communications in Partial Differential Equations 34.10 (2009): 1228-1250.
- [4] M.J. Cáceres and J.A. Cañizo. "Close-to-equilibrium behaviour of quadratic reaction-diffusion systems with detailed balance." to appear in Nonlinear Analysis: TMA (2017).
- [5] L. Desvillettes, K. Fellner and B.Q. Tang, "Trend to equilibrium for reaction-diffusion systems arising from complex balanced chemical reaction networks", accepted in SIAM Journal on Mathematical Analysis.
- [6] L. Desvillettes, K. Fellner, M. Pierre and J. Vovelle. Global existence for quadratic systems of reaction-diffusion. Advanced Nonlinear Studies, 7(3), (2007) 491-511.
- [7] M. Feinberg, "Lectures on Chemical Reaction Networks", University of Wisconsin-Madison, 1979, <https://crnt.osu.edu/LecturesOnReactionNetworks>.
- [8] M. Feinberg, "Chemical reaction network structure and the stability of complex isothermal reactors. I. The deficiency zero and deficiency one theorems", Chem. Eng. Sci., 42 (1987), pp. 2229-2268.
- [9] K. Fellner and E.H. Laamri. "Exponential decay towards equilibrium and global classical solutions for nonlinear reaction-diffusion systems" J. Evol. Equ. (2016) 16: 681.
- [10] K. Fellner and B.Q. Tang, "Convergence to equilibrium for renormalised solutions to nonlinear chemical reaction-diffusion systems", Preprint.
- [11] K. Fellner, E. Latos and T. Suzuki. "Global classical solutions for mass-conserving, (super)-quadratic reaction-diffusion systems in three and higher space dimensions." Discrete and Continuous Dynamical Systems - Series B, 2016, 21, 3441-3462.

- [12] J. Fischer. "Global existence of renormalized solutions to entropy-dissipating reaction-diffusion systems." *Archive for Rational Mechanics and Analysis* 218.1 (2015): 553-587.
- [13] A. Glitzky and R. Hunlich. "Global estimates and asymptotics for electro reaction diffusion systems in heterostructures." *Applicable Analysis* 66.3-4 (1997): 205-226.
- [14] T. Goudon and A. Vasseur. "Regularity analysis for systems of reaction-diffusion equations." *Annales scientifiques de l'Ecole normale supérieure*. Vol. 43. No. 1. Elsevier, 2010.
- [15] O.A. Ladyzhenskaya and N.N. Ural'tseva. "Linear and quasilinear equations of elliptic type." (1973): 576.
- [16] F. Horn and R. Jackson. "General mass action kinetics." *Archive for rational mechanics and analysis* 47.2 (1972): 81-116.
- [17] A. Mielke. "Uniform exponential decay for reaction-diffusion systems with complex-balanced mass-action kinetics". WIAS preprint 2326.
- [18] M. Pierre. "Weak solutions and supersolutions in  $L^1$  for reaction-diffusion systems." *Nonlinear Evolution Equations and Related Topics*. Birkhuser Basel, 2003. 153-168.
- [19] M. Pierre, D. Schmitt. "Blowup in reaction-diffusion systems with dissipation of mass." *SIAM review* 42.1 (2000): 93-106.
- [20] M. Pierre, "Global existence in reaction-diffusion systems with control of mass: a survey", *Milan Journal of Mathematics* 78.2 (2010): 417-455.
- [21] F. Rothe. "Global solutions of reaction-diffusion systems." *Lecture Notes in Mathematics*, Vol. 1072, Springer 1984.
- [22] D. Siegel and M.D. Johnston, "Linearization of complex balanced chemical reaction systems" Preprint 2008.
- [23] B.Q. Tang, "Global classical solutions to reaction-diffusion systems in one and two dimensions". arXiv:1702.07003.

INSTITUTE FOR MATHEMATICS AND SCIENTIFIC COMPUTING,, UNIVERSITY OF GRAZ, AUSTRIA  
*E-mail address:* quoc.tang@uni-graz.at